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Interpolation by Smooth Functions under Restrictions on the Derivatives

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INTRODUCTION

Interpolation by convex splines has recently attracted some attention, cf. McAllister *et al.* [4], McAllister and Roulier [5], Passow and Roulier [7] and Pence [8]. In this paper it is shown that the smoothest convex interpolating function to convex data is a cubic spline. The proof of this result is based on the degree theory of mappings in finite dimensional Euclidian space, cf. Ortega and Rheinboldt [6, Chap. 6]. The similar problem of finding the smoothest monotone interpolating function to monotone data was solved in Hornung [3].

1. THE PROBLEM AND MAIN RESULTS

Let a set $X = \{x_1, \dots, x_n\}$ of fixed data points in the interval $[a, b]$ with $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$, $n \geqslant 1$, and some boundary data $z_0, z_{n+1} \in \mathbb{R}$ be given. The space of functions on $[a, b]$ having square integrable derivatives of the k-th order is denoted by $H^k(a, b)$. It is a Hilbert space with norm

$$\|u\| = \left(\sum_{j=0}^{k} \|D^j u\|^2_{L^2(a,b)}\right)^{1/2}.$$

(1.1) DEFINITION. For an integer number k with $2 \leqslant k \leqslant n-2$, a real number γ, and a vector $z = (z_1, \dots, z_n)^T \in \mathbb{R}^n$ of interpolation data, a function u is called admissible, if $u \in H^k(a, b)$ satisfies $u(x_i) = z_i$ for $i = 0, \dots, n+1$ and $D^k u \geqslant \gamma$ a.e. on $[a, b]$. The set of all admissible functions is denoted by M_γ^k.

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(1.2) PROBLEM. A minimizer of the functional $F(u) = \frac{1}{2}\int_a^b (D^k u(x))^2\, dx$ on the set M_γ^k of admissible functions is called a solution of problem A_γ^k.

(1.3) DEFINITION. (a) We denote by P^k the space of all splines p on $[a, b]$ of degree $k - 1$ with knots in X, which satisfy $D^j p(x_i) = 0$ for $i = 0$, $i = n - 1$ and $j = 0,..., k - 2$.

(b) For $p \in P^k$ and $\alpha \in \mathbb{R}$ we set

$$p_\alpha(x) = \begin{cases} p(x), & \text{if } p(x) \geqslant \alpha, \\ \alpha, & \text{otherwise.} \end{cases}$$

(c) We say that a function u satisfies condition C_γ^k, if there is a $p \in P^k$ with $D^k u = p_\gamma$ on $[a, b]$.

THEOREM 1. *If for a given data vector $z \in \mathbb{R}^n$ there is an admissible function, problem A_γ^k has a unique solution. For an admissible function to solve problem A_γ^k condition C_γ^k is sufficient.*

(1.4) DEFINITION. (a) For $z \in \mathbb{R}^n$ we denote by $\Delta(z) = q = (q_1 ,..., q_n)^T \in \mathbb{R}^n$ the vector of second difference quotients

$$q_i = \frac{2}{x_{i+1} - x_{i-1}} \left(\frac{z_{i+1} - z_i}{x_{i+1} - x_i} - \frac{z_i - z_{i-1}}{x_i - x_{i-1}}\right).$$

(b) A data vector $z \in \mathbb{R}^n$ is called γ-convex, if $\Delta(z)$ is an element of the set $Q_\gamma = \{(q_1 ,..., q_n)^T \in \mathbb{R}^n : q_i > \gamma \text{ for } i = 1,..., n\}$.

THEOREM 2. *Let $z \in \mathbb{R}^n$ be a γ-convex data vector. Then problem A_γ^2 has a unique solution. For an admissible function to solve problem A_γ^2 condition C_γ^2 is necessary and sufficient. The solution is a cubic spline having at most*

$$m(n) = \begin{cases} 3\,\dfrac{n+1}{2}, & \textit{if } n \textit{ odd}, \\[2ex] 3\,\dfrac{n}{2}, & \textit{if } n \textit{ even}, \end{cases}$$

knots in (a, b): it depends continuously on the data z.

Since the interpolating natural spline of degree $2k - 1$ with knots in X satisfies condition C_β^k if $-\beta$ is sufficiently large, theorems 1 and 2 are generalizations of the well known minimal properties of natural splines. First we prove theorem 1.

(1.5) LEMMA. *The functional F is Fréchet-differentiable on $H^k(a, b)$,* *strictly convex on M_γ^k and coercive over M_γ^k, i.e. $\lim F(u) = +\infty$ holds for $\|u\| \to \infty$ and $u \in M_\gamma^k$.*

Proof. If

$$\langle F'(u), \varphi \rangle = \int_a^b D^k u(x)\, D^k \varphi(x)\, dx,$$

we have

$$F(u + \varphi) - F(u) - \langle F'(u), \varphi \rangle = \frac{1}{2} \int_a^b (D^k \varphi(x))^2\, dx = o(\|\varphi\|)$$

for $u, \varphi \in H^k(a, b)$, i.e. F' is the Fréchet-differential of F. For any $u, v \in M_\gamma^k$, $u \neq v$ we have

$$\langle F'(u) - F'(v), u - v \rangle = \int_a^b (D^k(u - v)(x))^2\, dx;$$

since $u(x_i) = v(x_i) = z_i$ for $i = 0,..., n + 1$, and $k \leqslant n + 2$, this integral is positive. Hence F is strictly convex on M_γ^k, cf. Ekeland/Temam [2, Chap. I, Prop. 5.4 and 5.5]. Let $\bar{u}$ be a polynomial of degree $k - 1$, which interpolates exactly k data (x_i, z_i), $i \in I \subset \{0,..., n + 1\}$. Then for $\|u\| \to \infty$ we have $\|u - \bar{u}\| \to \infty$. On the subspace U of $H^k(a, b)$ consisting of those functions $\tilde{u}$, for which $\tilde{u}(x_i) = 0$ for $i \in I$, the norm

$$\left(\int_a^b (D^k \tilde{u}(x))^2\, dx \right)^{1/2}$$

is equivalent to the norm induced from $H^k(a, b)$. Therefore $\|u\| \to \infty$ implies

$$F(u) = \frac{1}{2} \int_a^b (D^k u(x))^2\, dx = \frac{1}{2} \int_a^b (D^k(u - \bar{u})(x))^2\, dx \to \infty,$$

i.e. F is coercive over M_γ^k.

Proof of theorem 1. Since M_γ^k is nonvoid, closed and convex, existence and uniqueness of a minimizer follow from (1.5), cf. Ekeland/Temam [2, Chap. II, Prop. 1.2]. Let $u \in M_\gamma^k$ satisfy condition C_γ^k, and $p \in P^k$ be chosen according to (1.3c). Then from the proof of (1.5) we have

$$\langle F'(u), \varphi \rangle = \int_a^b p_\gamma(x)\, D^k \varphi(x)\, dx$$

for any $\varphi \in H^k(a, b)$. If p is extended by zero outside $[a, b]$, and $\lambda_i = D^{k-1} p(x_i + 0) - D^{k-1} p(x_i - 0)$ for $i = 0,..., n + 1$, integration by parts yields

$$\int_a^b p(x)\, D^k \varphi(x)\, dx = (-1)^k \sum_{i=0}^{n+1} \lambda_i \varphi(x_i),$$

hence

$$\langle F'(u), \varphi\rangle = (-1)^k \sum_{i=0}^{n-1} \lambda_i \varphi(x_i) + \int_a^b (p_\gamma(x) - p(x))\, D^k\varphi(x)\, dx$$

holds for $\varphi \in H^k(a, b)$. For $v \in M_\gamma^k$ we have $v(x_i) = u(x_i)$ for $i = 0,\dots, n-1$, therefore

$$\langle F'(u), v - u\rangle = \int_a^b (p_\gamma(x) - p(x))(D^k v(x) - p_\gamma(x))\, dx.$$

It is easy to see that this integral is nonnegative. Since $p_\gamma - p \geq 0$, we have to consider only an $x \in [a, b]$, for which $p_\gamma(x) - p(x) > 0$, i.e. $p(x) < \gamma$ and $p_\gamma(x) = \gamma$. From $D^k v(x) \geq \gamma$ for almost all $x \in [a, b]$ we deduce

$$\langle F'(u), v - u\rangle \geq 0$$

for any $v \in M_\gamma^k$. Therefore, u is a minimizer of F on M_γ^k, cf. Ekeland and Temam [2, Chap. II, Prop. 2.1].

The remainder of the paper is devoted to the proof of theorem 2. The first step is the demonstration that γ-convexity of a data vector z implies the existence of an admissible function.

(1.6) LEMMA. *If $z \in \mathbb{R}^n$ is γ-convex, then there is a function $v \in C^\infty[a, b]$ with $v(x_i) = z_i$ for $i = 0,\dots, n-1$ and $D^2 v > \gamma$ on $[a, b]$: the set M_γ^2 is nonvoid.*

Proof. Let $q = (q_1, \dots, q_n)^T = \Delta(z)$ and $\bar\gamma$ be chosen such that $q_i > \bar\gamma > \gamma$ for $i = 1,\dots, n$. Then we define

$$s_{i-1/2} = \frac{z_{i-1} - z_i}{x_{i-1} - x_i}, \qquad i = 0,\dots, n,$$

and

$$\kappa_i = \frac{s_{i+1/2} - s_{i-1/2}}{x_{i+1} - x_{i-1}} - \frac{\bar\gamma}{2}, \qquad i = 1,\dots, n.$$

Since $q \in Q_{\bar\gamma}$, we have $\kappa_i > 0$. For

$$\kappa_0 = \kappa_{n-1} = 1,$$

$$\sigma_0 = s_{1/2} - (x_1 - x_0)\left(\frac{\bar\gamma}{2} + 1\right),$$

$$\sigma_i = \frac{1}{x_{i+1} - x_{i-1}}((x_{i+1} - x_i) s_{i-1/2} + (x_i - x_{i-1})\, s_{i+1/2}), \quad i = 1,\dots, n,$$

and

$$\sigma_{n+1} = s_{n+1/2} + (x_{n+1} - x_n)\left(\frac{\bar{\gamma}}{2} - 1\right)$$

we obtain

$$\frac{\sigma_{i-1} - \sigma_i}{x_{i-1} - x_i} - \bar{\gamma} = \kappa_i - \kappa_{i-1} > 0$$

for all $i = 0,..., n$. For any $\eta \in (0, 1)$ a nondecreasing function $\psi_\eta \in C^\infty[0, 1]$ can be chosen such that $\psi_\eta(0) = 0$, $\psi_\eta(1) = 1$, $D^j\psi_\eta(0) = D^j\psi_\eta(1) = 0$ for all $j = 1, 2,...$, and $\int_0^1 \psi_\eta(t)\, dt = \eta$. We define

$$\eta_i = \frac{\kappa_i}{\kappa_i - \kappa_{i-1}},$$

$$w_i(x) = (\sigma_{i-1} - \sigma_i - \bar{\gamma}(x_{i-1} - x_i))\, \psi_{\eta_i}\left(\frac{x - x_i}{x_{i-1} - x_i}\right) + \bar{\gamma}(x - x_i),$$

and

$$v_i(x) = z_i + \sigma_i(x - x_i) + \int_{x_i}^{x} w_i(\xi)\, d\xi$$

for $i = 0,..., n$. Now we have $v_i \in C^\infty[x_i, x_{i-1}]$ and $v_i(x_l) = z_l$, $Dv_i(x_l) = \sigma_l$, $D^2v_i(x_l) = \bar{\gamma}$, $D^jv_i(x_l) = 0$ for $j = 3, 4,...$, $l = i, i+1$ and $D^2v_i \geqslant \bar{\gamma}$ on $[x_i, x_{i-1}]$. Therefore the function

$$v(x) = v_i(x) \text{ for } x \in [x_i, x_{i-1}]$$

has the desired properties.

In the proof of theorem 2 the necessity of condition C_γ^2 remains to be shown. As a preparation we reformulate this condition as an operator equation.

(1.7) DEFINITION. (a) Let $r_0 = r_{n-1} = 0$. For a vector $r = (r_1,..., r_n)^T \in \mathbb{R}^n$ we denote by $\Pi(r) = p \in P^2$ the polygonal function on $[a, b]$ with knots in X, for which $p(x_i) = r_i$ holds.

(b) Let G be Green's function for the differential operator D^2 on $[a, b]$ with boundary conditions $u(a) = u(b) = 0$, i.e.

$$G(x, t) = \frac{1}{b-a}\begin{cases}(x-b)(t-a), & \text{if } a \leqslant t \leqslant x \leqslant b\\ (x-a)(t-b), & \text{if } a \leqslant x < t \leqslant b.\end{cases}$$

Then for $p \in P^2$ and $x \in \mathbb{R}$ we denote by $\Gamma_x(p) = u$ the function on $[a, b]$ defined by

$$u(x) = \frac{1}{b-a}((b-x)\, z_0 + (x-a)\, z_{n-1}) - \int_a^b G(x, t)\, p_x(t)\, dt.$$

(c) For a function u on $[a, b]$ let $\Lambda(u) = z$ be the vector $(z_1, \ldots, z_n)^T \in \mathbb{R}^n$ with $z_i = u(x_i)$ for $i = 1, \ldots, n$.

(d) For $\gamma \in \mathbb{R}$ we define $S_\gamma = \Gamma_\gamma \circ \Pi$ and $T_\gamma = \Delta \circ \Lambda \circ S_\gamma$.

(1.8) COROLLARY. *If $z \in \mathbb{R}^n$ is γ-convex, $q = \Delta(z)$, $r \in \mathbb{R}^n$, and $T_\gamma(r) = q$, then $u = S_\gamma(r)$ is the solution of problem A_γ^2.*

Proof. Since $\Delta : \mathbb{R}^n \to \mathbb{R}^n$ is a regular affine mapping, we have $z = \Lambda \circ S_\gamma(r) = \Lambda(u)$, i.e. $u(x_i) = z_i$ for $i = 0, \ldots, n - 1$. For $p = \Pi(r)$ we obtain from (1.7b, d) $u = \Gamma_\gamma(p)$ and $D^2 u = p_\gamma \geq \gamma$ on $[a, b]$. Evidently u is admissible and condition C_γ^2 is satisfied. According to theorem 1 the function u solves problem A_γ^2.

2. THE HOMOTOPY

In this paragraph we study the operator family T_γ. We begin with a well known representation of the second difference quotient.

(2.1) LEMMA. (a) *If*

$$\omega_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{for } x_{i-1} \leq x \leq x_i, \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{for } x_i \leq x \leq x_{i+1}, \end{cases}$$

$r \in \mathbb{R}^n$, $q = T_\gamma(r)$, and $p = \Pi(r)$, we have

$$q_i = \frac{2}{x_{i+1} - x_{i-1}} \int_{x_{i-1}}^{x_{i+1}} p_\gamma(x)\, \omega_i(x)\, dx.$$

(b) *If $r_j \geq \gamma$ for $j = i - 1, i, i + 1$, then*

$$q_i = \frac{1}{3} \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} r_{i-1} + \frac{2}{3} r_i + \frac{1}{3} \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} r_{i+1}.$$

Proof. Since

$$q_i = 2\Delta^2(x_{i-1}, x_i, x_{i+1})\, \Gamma_\gamma(p),$$

formula (a) follows from

$$\Delta_t^2(x_{i-1}, x_i, x_{i+1})(x - t)_+ = \frac{\omega_i(t)}{x_{i+1} - x_{i-1}}$$

according to Peano's theorem on the representation of linear functionals, cf. Werner/Schaback [9, example 4.5]. Formula (b) is a direct consequence of (a).

(2.2) LEMMA. *Let* $r \in \mathbb{R}^n$, $T_\alpha(r) = q$, *and* $-\rho \leqslant -\beta \leqslant \alpha \leqslant \gamma \leqslant \sigma$, $-\rho < \sigma$.

(a) *If* $r_i = -\rho$ *and* $r_{i-1}, r_{i+1} \leqslant \sigma$, *then*

$$q_i - \gamma \leqslant c_1 = \frac{(\sigma - \gamma)^3}{3(\sigma + \rho)^2}.$$

(b) *If* $r_i = \sigma$ *and* $r_{i-1}, r_{i+1} \geqslant -\rho$, *then*

$$q_i \geqslant c_2 = \frac{2\sigma^3 + 3\rho\sigma^2 - 3\beta^2\rho - \beta^3 - 3\beta\rho^2}{3(\sigma - \rho)^2}.$$

Proof. From (2.1) we have $q_i = 1/(x_{i+1} - x_{i-1})\,(q_i^- + q_i^+)$ with

$$q_i^- = 2\int_{x_{i-1}}^{x_i} p_\alpha(x)\,\omega_i(x)\,dx \qquad \text{and} \qquad q_i^+ = 2\int_{x_i}^{x_{i+1}} p_\alpha(x)\,\omega_i(x)\,dx.$$

(a) First we consider q_i^+. If we define $x = x_i + (x_{i-1} - x_i)\,\tau$, $\tau_0 = (\rho - \gamma)/(\rho - \sigma)$, and

$$\bar{p}(\tau) = \begin{cases} \gamma, & \text{if } 0 \leqslant \tau \leqslant \tau_0, \\ -\rho + (\sigma + \rho)\tau, & \text{if } \tau_0 < \tau \leqslant 1, \end{cases}$$

the assumptions on r imply $p_\alpha(x) \leqslant \bar{p}(\tau)$ for $\tau \in [0, 1]$. Since $\omega_i \geqslant 0$, we obtain the inequality

$$\begin{aligned} \frac{1}{x_{i-1} - x_i}\, q_i^+ &\leqslant \frac{2}{x_{i-1} - x_i} \int_{x_i}^{x_{i-1}} \bar{p}(\tau)(1 - \tau)\, dx \\ &= 2\int_0^1 \bar{p}(\tau)(1 - \tau)\, d\tau \\ &= 2\left(\int_0^{\tau_0} \gamma(1 - \tau)\, d\tau + \int_{\tau_0}^1 (-\rho + (\sigma - \rho)\tau)(1 - \tau)\, d\tau\right) \\ &= \frac{\sigma - 2\rho}{3} + \frac{(\rho + \gamma)^2}{\rho - \sigma} - \frac{1}{3}\frac{(\rho - \gamma)^3}{(\rho - \sigma)^2} = c_1 - \gamma. \end{aligned}$$

In a similar way we get

$$\frac{1}{x_i - x_{i-1}}\, q_i^- < c_1 - \gamma.$$

This yields

$$q_i = \frac{1}{x_{i+1} - x_{i-1}}((x_{i+1} - x_i) + (x_i - x_{i-1}))(c_1 + \gamma) = c_1 + \gamma.$$

(b) First we consider q_i^+. If we define $x = x_i + (x_{i+1} - x_i)\,\tau$, $\tau_0 = (\sigma + \beta)/(\sigma + \rho)$,

$$\bar{p}(\tau) = \begin{cases} \sigma - (\sigma + \rho)\tau, & \text{if } 0 \leq \tau \leq \tau_0 \\ -\beta, & \text{if } \tau_0 \leq \tau \leq 1, \end{cases}$$

the assumptions on r imply $p_x(x) \geq \bar{p}(\tau)$ for $\tau \in [0, 1]$. Since $\omega_i \geq 0$, we obtain the inequality

$$\frac{1}{x_{i+1} - x_i}\, q_i^+ \geq \frac{2}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \bar{p}(\tau)(1 - \tau)\, dx$$

$$= 2\left(\int_0^{\tau_0} (\sigma - (\sigma + \rho)\tau)(1 - \tau)\, d\tau + \int_{\tau_0}^1 - \beta(1 - \tau)\, d\tau\right)$$

$$= \frac{(\sigma + \beta)^2}{\sigma + \rho} - \frac{1}{3}\frac{(\sigma + \beta)^3}{(\sigma + \rho)^2} - \beta = c_2 .$$

In a similar way we get

$$\frac{1}{x_{i+1} - x_i}\, q_i^- \geq c_2 .$$

This yields $q_i \geq c_2$.

(2.3) LEMMA. *If $-\beta < \gamma < \beta$ and $\epsilon > 0$, there are numbers $\rho, \sigma > \beta$ such that for c_1, c_2 in* (2.2) *the inequalities*

$$c_1 < \epsilon \quad \textit{and} \quad c_2 > \frac{1}{\epsilon}$$

are valid.

Proof. If $\rho = t^5$ and $\sigma = t^3$, we obtain from (2.2)

$$c_1 = \frac{(t^3 - \gamma)^3}{3(t^5 + t^3)^2} \to 0$$

and

$$c_2 = \frac{2t^9 + 3t^{11} + 3\beta^2 t^5 - \beta^3 - 3\beta t^{10}}{3(t^5 + t^3)^2} \to \infty$$

for $t \to \infty$.

(2.4) DEFINITION. For $\rho, \sigma > 0$ let

$$K(\rho, \sigma) = \{(r_1, \ldots, r_n)^T \in \mathbb{R}^n \quad -\rho < r_i < \sigma \text{ for } i = 1, \ldots, n\}$$

and $\partial K(\rho, \sigma)$ be the boundary of the cube $K(\rho, \sigma)$.

(2.5) COROLLARY. *If $q \in Q_\gamma$ and $-\beta < \gamma < \beta$, there are numbers $\rho, \sigma \geqslant \beta$ such that*

$$q \notin T_\alpha(\partial K(\rho, \sigma))$$

for all $\alpha \in [-\beta, \gamma]$.

Proof. Since $q_i > \gamma$, there is an $\epsilon > 0$ such that $q_i - \gamma > \epsilon$ and $q_i < (1/\epsilon)$ for $i = 1, \ldots, n$. If $\rho, \sigma \geqslant \beta$ are chosen according to (2.3) and $r \in \partial K(\rho, \sigma)$, we have $r_i = -\rho$ or $r_i = \sigma$ for some $i = 1, \ldots, n$. If $q = T_\alpha(r)$, then (2.2a) implies $\epsilon < q_i - \gamma \leqslant c_1 \leqslant \epsilon$ in the first case and (2.2b) implies $(1/\epsilon) > q_i > c_2 \geqslant (1/\epsilon)$ in the second. Thus we get a contradiction in both cases.

3. THE DEGREE

In this paragraph we show that for $q \in Q_\gamma$ the degree $\deg(T_\gamma, K(\rho, \sigma), q)$ of the mapping T_γ is nonzero, if the cube $K(\rho, \sigma)$ is chosen appropriately. From this, theorem 2 is easily deduced.

(3.1) LEMMA. *If $\gamma \in \mathbb{R}$ and $q \in Q_\gamma$, there is a number $\beta > |\gamma|$ such that*

$$deg(T_{-\beta}, K(\rho, \sigma), q) \neq 0$$

for all $\rho, \sigma \geqslant \beta$.

Proof. Let $z = (z_1, \ldots, z_n)^T = \Delta^{-1}(q)$ and u^* be the natural cubic spline on $[a, b]$ with knots in X, which interpolates the data (x_i, z_i), $i = 0, \ldots, n + 1$. Let $r^* = (r_1^*, \ldots, r_n^*)$, $r_i^* = D^2u^*(x_i)$ for $i = 1, \ldots, n$, and $\beta > \max_{1 \leqslant i \leqslant n} | r_i^* |$. Then for $\alpha = -\beta$ we have $S_\alpha(r^*) = u^*$ and $T_\alpha(r^*) = q$. Since (2.1b) applies for $r \in K(\beta, \sigma)$, the mapping T_α is linear on $K(\beta, \sigma)$. The matrix corresponding to T_α is diagonally dominant, therefore T_α is regular and r^* is the unique solution r of the equation $T_\alpha(r) = q$ in the cube $K(\beta, \sigma)$. From an elementary property of the degree we obtain

$$\deg(T_\alpha, K(\beta, \sigma), q) \in \{+1, -1\},$$

cf. Ortega and Rheinboldt [6, 6.1.2]. It follows that $q \notin T_\alpha(\overline{K(\rho, \sigma)} - K(\beta, \sigma))$. For otherwise, we have two solutions of problem A_α^2 according to (1.8), namely $u^* = S_\gamma(r^*)$ with $r^* \in K(\beta, \sigma)$, i.e., $-\beta < r_i^* < \sigma$ for $i = 1, \dots, n$, and on the other hand $u = S_\gamma(r)$ for some $r \in \overline{K(\rho, \sigma)} - K(\beta, \sigma)$, i.e., $r_i \leq -\beta < \gamma$ or $r_i \geq \sigma > \beta$ for some $i = 1, \dots, n$. Consequently we have for $p^* = \Pi(r^*)$ and $p = \Pi(r)$

$$D^2u^* = p^*_\gamma \neq p_\gamma = D^2u,$$

so u^* and u are distinct, which contradicts theorem 1. From the excision theorem, cf. Ortega and Rheinboldt [6, 6.2.8], it then follows

$$\deg(T_\gamma, K(\rho, \sigma), q) = \deg(T_\gamma, K(\beta, \sigma), q) \neq 0.$$

(3.2) COROLLARY. *If* $\gamma \in \mathbb{R}$ *and* $q \in Q_\gamma$, *there are numbers* $\rho, \sigma > \gamma$, *such that*

$$\deg(T_\gamma, K(\rho, \sigma), q) \neq 0;$$

equation $T_\gamma(r) = q$ *has a solution* $r \in K(\rho, \sigma)$.

Proof. Let $\beta > \gamma$ be chosen according to (3.1) and $\rho, \sigma > \beta$ as in (2.5). Since the mapping $T : [-\beta, \gamma] \times \overline{K(\rho, \sigma)} \to \mathbb{R}^n$ is continuous, we can apply the theorem on the homotopy invariance of the degree, cf. Ortega/Rheinboldt [6, 6.2.2]. From (2.5) and (3.1) we deduce that the degree is nonzero. The solvability of the operator equation follows from Kronecker's theorem, cf. Ortega/Rheinboldt [6, 6.3.1].

Proof of theorem 2. From theorem 1 and (1.6) we have the existence and uniqueness of a solution and the sufficiency of condition C_γ^2. For the demonstration of the necessity of C_γ^2, let $q = \Delta(z) \in Q_\gamma$ and u be the solution of A_γ^2. From (3.2) there is a solution of the equation $T_\gamma(r) = q$. Therefore, (1.8) implies that $\tilde{u} = S_\gamma(r)$ solves problem A_γ^2. Since this solution is unique, we have $\tilde{u} = u$. For $p = \Pi(r) \in P^2$ we have $\Gamma_\gamma(p) = u$, i.e. $D^2u = p_\gamma$. This means that C_γ^2 is satisfied. Evidently p_γ is a polygonal function on $[a, b]$ having at most $m(n)$ knots. Therefore, u is a cubic spline. From (2.1) it is easily seen that the solution of the equation $T_\gamma(r) = q$ is in the open set R_γ of those $r \in \mathbb{R}^n$ which satisfy $r_{i-1} < \gamma$ or $r_{i+1} < \gamma$ if $r_i \leqq \gamma$, $i = 1, \dots, n$. Since the spline u is unique, the function $p_\gamma = D^2u$ and the vector r, $r_i = p(t_i)$ are uniquely determined. Therefore, $T_\gamma : R_\gamma \to Q_\gamma$ is a continuous one-to-one mapping. The domain invariance theorem, cf. Deimling [1, Theorem 11.3], implies the continuity of T_γ^{-1}. Hence, $u = S_\gamma \circ T_\gamma^{-1} \circ \Delta(z)$ depends continuously on z.

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